Confocal surfaces and integrable billiards on the sphere and in the Lobachevsky space

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Abstract. *Billiard problems for the domains on the sphere and hyperbolic space bounded by the corresponding conical sections are investigated. It is shown that these discrete systems are integrable and correspond to the translations on the Jacobi varieties of certain hyperelliptic curves. The explicit formulas in terms of θ-functions are exhibited. The consideration is based on the factorization method, developed recently by J. Moser and the author.*

INTRODUCTION

It is a well-known geometrical fact that the tangent line to the geodesic on the ellipsoid in euclidean space is touching the set of confocal quadrics which is fixed for a given geodesic (see, for example [1]). In the limit when the length of one of the axes tends to zero we have a billiard problem in the domain bounded by the ellipsoid of one less dimension. It means that the billiard's trajectories possess the same geometrical property, which leads to the number of the involutive integrals sufficient for the complete integrability in Liuoville sense (see [2, 3]).

Nevertheless, it seems to be difficult to use this limit procedure for the integration of the billiard's dynamics because of the singular nature of this limit.

In the recent paper [4] J. Moser and the author proposed the method for

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the integration of discrete systems and applied it to some discrete versions of the classical integrable systems. In particular, the ellipsoidal billiard dynamics can be explicitly described by using this technique, which is based on the factorizations of the matrix polynomials [4].

It turns out that the methods of [4] are applicable to certain billiard problems in the domains in the sphere and in the Lobachevsky space which can be considered as natural generalizations of the euclidean ellipsoidal domains.

The aim of this paper is to explain how these problems can be integrated by using the factorization procedure. The related geometrical and mechanical problems in Lobachevsky space are also discussed.

In §1 the surfaces in the sphere and in the Lobachevsky space, which are the analogues of the euclidean ellipsoids are defined. In the case of the sphere these surfaces, which are called the sphere-conical sections or the spherical conics, are the subject of geometrical investigations, originating with Chasles (see [1, 5]). These surfaces are defined as the intersections of the sphere and the cone where the vertex is at the center of the sphere. The coordinates, connecting with the appropriately defined confocal family, were used by C. Neumann in 1859 for the integration of the system, describing the motion of the mass point on the sphere under the influence of the force with the quadratic potential [6].

The hyperbolic case does not seem to have been discussed in the literature although the definition is quite similar. Instead of the sphere one can consider the hyperboloid in the pseudoeuclidean space $\mathbb{R}^{n,1}$, one sheet of which, with the induced metric, is the well-known model of the Lobachevsky space. The suitable cone intersects this sheet in a compact closed surface, which we call the H-ellipsoid. We show that the naturally defined confocal family of H-conical sections possesses all geometrical properties of the euclidean and spherical one. It is not completely obvious because of the reality conditions.

In §2 the factorization method [4] is explained and applied for the integration of the billiard problem in the domain on the sphere and the Lobachevsky space bounded by conical sections. As a consequence we show that these dynamical systems correspond to the translation on the Jacobi varieties of the certain hyperelliptic curves and give the explicit formulas for the general solutions in terms of θ-functions. In §4 we discuss some geometrical corollaries.

In the Appendices the hyperbolic analogues of two classical integrable systems are considered. Appendix A is devoted to the geodesic flow on the H-ellipsoid in Lobachevsky space. We present the Lax representation with the spectral parameter of the dynamics. This representation is crucial for the integrability. In the euclidean case such representation is the consequence of Moser's results [6].

In Appendix B we consider the hyperbolic version of C. Neumann's system, which can be viewed as the anisotropic harmonic oscillator in the Lobachevsky space. We give the interpretation of the motion in terms of the eigenfunctions of the finite-gap Schrödinger operators

$$L = -\frac{d^2}{dt^2} + u(t).$$

This result is the hyperbolic generalization of the Moser-Trubowitz discovery (see [2, 7-9]). As a consequence the θ-functional formulas for the solutions are given.

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1. CONFOCAL FAMILIES IN THE SPHERE AND LOBACHEVSKY SPACE

We will consider the real $(n + 1)$-dimensional vector space V with symmetric bilinear form

$$(1) \qquad \langle x, y\rangle = J_0 x_0 y_0 + J_1 x_1 y_1 + \ldots + J_n x_n y_n$$

which will be supposed to be nondegenerate. Such form determines the isomorphism $J: V \to V^*$ of V and the dual space V^* by the formula

$$(2) \qquad \langle x, y\rangle = (x, Jy) = (x, y^*)$$

where the right hand side is the value of the linear function $y^* = Jy \in V^*$ at the point $x \in V$.

Actually only two cases will be of interest: $J_0 = J_1 = \ldots = J_n = 1$ and $J_0 = -1, J_1 = J_2 = \ldots = J_n = 1$ corresponding to the euclidean space $V = \mathbb{R}^{n+1}$ and to the $(n + 1)$-dimensional Minkovski space $V = \mathbb{R}^{n,1}$.

In the last case the equation

$$(3) \qquad \langle x, x\rangle = -x_0^2 + x_1^2 + \ldots + x_n^2 = -1$$

determines a two-sheeted hyperboloid.

One sheet of that hyperboloid with the induced metric which is positive definite is the well-known model of the n-dimensional Lobavhevsky space H. The straight lines or geodesic in this model are the sections of the hyperboloid by the planes passing through the origin in $\mathbb{R}^{n,1}$ (see, for example, [10]).

In the euclidean case we have the unit sphere

$$(4) \qquad \langle x, x\rangle = x_0^2 + \ldots + x_n^2 = 1$$

which is after the identification of x with $-x$ the model of elliptic or spherical geometry. The reason for the identification is that any two straight lines, which are now the great circles, should have no more than one common point.

The following definition is possible in any pseudoeuclidean space V (compare with [10, 11]). Any quadric Q in V can be represented as

$$Q(x) = \langle x, Ax\rangle \tag{5}$$

for some selfadjoint operator $A : V \to V$:

$$\langle Ax, y\rangle = \langle x, Ay\rangle.$$

DEFINITION. The pencil of quadric Q_λ in V is called *pseudoeuclidean* if it has a form

$$Q_\lambda(x) = Q(x) + \lambda \langle x, x\rangle = \langle (A + \lambda I)x, x\rangle. \tag{6}$$

The dual family

$$\hat{Q}_\lambda(x) = \langle (A + \lambda I)^{-1} x, x\rangle$$

and the corresponding family of conics

$$\langle (A + \lambda I)^{-1} x, x\rangle = 0 \tag{7}$$

is called *confocal.* ■

In the euclidean case the intersections of these cones with the unit sphere form the family of the spherical conics (S-conics), which we also will call confocal. These surfaces form an orthogonal family and correspond to the orthogonal coordinate system on the sphere ("Elliptische Kugel Koordinaten"). These coordinates are defined as the roots $\lambda_1, \ldots, \lambda_n$ of the equation

$$\hat{Q}_\lambda(x) = \langle (A + \lambda I)^{-1} x, x\rangle = 0. \tag{8}$$

They were used by C. Neumann [6] for the integration of the mechanical system on the sphere with the quadratic potential $U(x) = \alpha \langle x, Ax\rangle$. He has shown that in these coordinates the separation of variables in the corresponding Hamilton-Jacobi equation takes place.

Figure 1 shows the projection of the confocal family for the sphere $S^2 \subset \mathbb{R}^3$ onto the x_0x_2 plane. The initial S-ellipse is defined by the equation

$$\frac{x_0^2}{a_0} + \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} = 0$$

with $a_0 > a_1 > 0 > a_2$. The foci correspond to the degenerate case $\lambda = a_1$ of the confocal curves

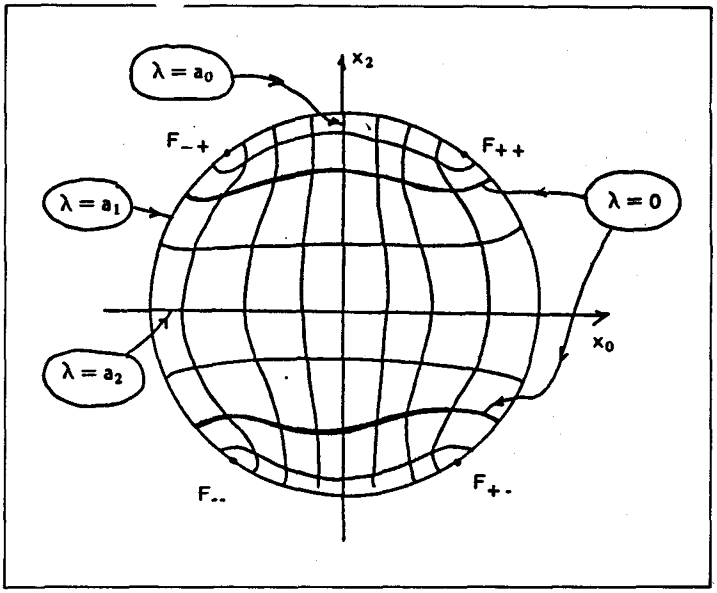

Fig. 1

$$\frac{x_0^2}{a_0-\lambda}+\frac{x_1^2}{a_1-\lambda}+\frac{x_2^2}{a_2-\lambda}=0$$

and have the coordinates

$$\left(\pm\sqrt{\frac{a_0-a_1}{a_0-a_2}},\ 0,\ \pm\sqrt{\frac{a_1-a_2}{a_0-a_2}}\right)$$

To the author's knowledge the hyperbolic case has not been discussed in the literature, therefore we consider it in more detail.

First of all the operator A, determining the quadric $Q(x) = \langle x, Ax\rangle$ for pseudo-euclidean space V, generally can not be diagonalized in the real. But in the case when the equation

(9) $$\hat{Q}(x)=\langle x, A^{-1}x\rangle = 0$$

determines the compact closed surface M^n in H the operator A are diagonalizable, as follows from the following result.

PROPOSITION 1. *Suppose that all points of the cone (9) satisfy the inequality $\langle x, x\rangle < 0$, which corresponds to the geometrical situation depicted in figure 2. Then the operator A^{-1} (and therefore A) is diagonal in some orthogonal coordinate system.*

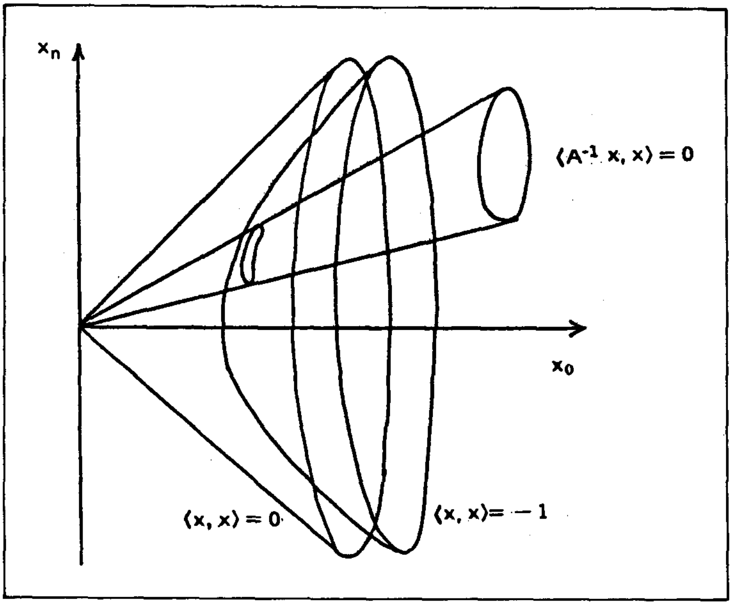

Fig. 2

Proof. Let us consider the function

$$f(x) = \frac{\langle A^{-1}x, x\rangle}{\langle x, x\rangle} .$$

This function vanishes on the surface M^n and has a maximum or minimum at some point x_0 of the domain bounded by M^n because of the compactness M^n. In this point we have $f'(x_0) = 0$ or

$$A^{-1}x_0 = \lambda_0 x_0, \lambda_0 = \frac{\langle A^{-1}x_0, x_0\rangle}{\langle x_0, x_0\rangle} \tag{10}$$

It means that x_0 is the eigenvector of the operator A^{-1}. In the orthogonal hyperspace $W = \{x \in V : \langle x, x_0\rangle = 0\}$ we have again two quadratic forms, which are the restrictions of $\hat{Q}(x)$ and $\langle x, x\rangle$. But now the second form is positive and one can use the well-known theorem of linear algebra to finish the proof. ∎

This result is the reason for the following definition.

DEFINITION. The compact surface M^n in the Lobachevsky space is called an *H-ellipsoid* if in some orthogonal coordinate system in $\mathbb{R}^{n,1}$ it is determined by the equation

$$(11) \qquad -\frac{x_0^2}{a_0}+\frac{x_1^2}{a_1}+\ldots+\frac{x_n^2}{a_n}=0$$

with $a_0 > a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$.

In the Klein model with the coordinates

$$\xi_i = \frac{x_i}{x_0} \quad (i = 1, \ldots, n)$$

an H-ellipsoid has the usual equation

$$(12) \qquad \frac{\xi_1^2}{b_1}+\frac{\xi_2^2}{b_2}+\ldots+\frac{\xi_n^2}{b_n}=1$$

for $b_i = a_i/a_0 < 1$.

The corresponding confocal family by the definition is as determined by

$$(13) \qquad -\frac{x_0^2}{a_0-\alpha}+\frac{x_1^2}{a_1-\alpha}+\ldots+\frac{x_n^2}{a_n-\alpha}=0.$$

For the general point $x \in H$ there exist precisely n H-conical sections, passing through x as follow from

PROPOSITION 2 *For given $x \in \mathbb{R}^{n,1}$ with $\langle x, x\rangle = -1$ and $x_0x_1 \ldots x_n \neq 0$ the equation (13) has n real roots satisfying the inequalities*

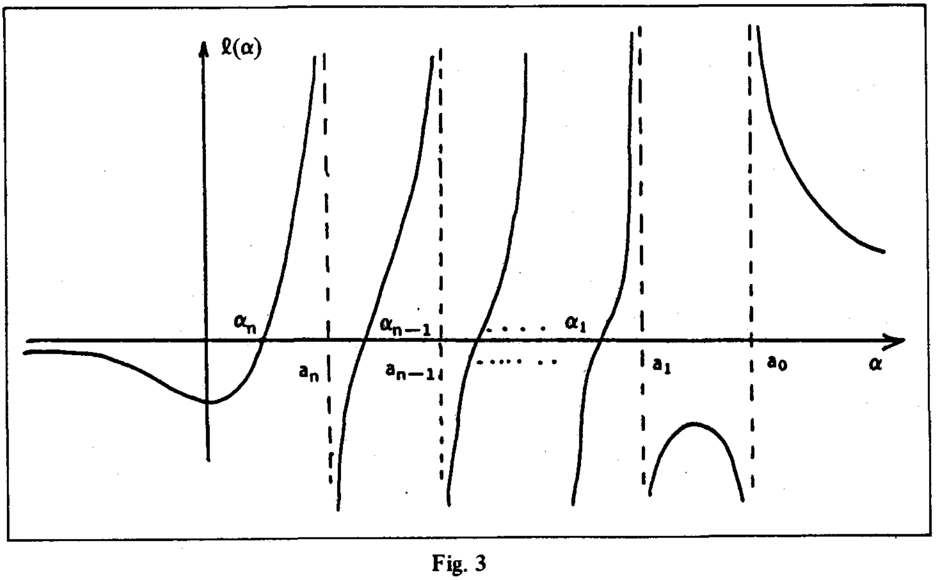

Fig. 3

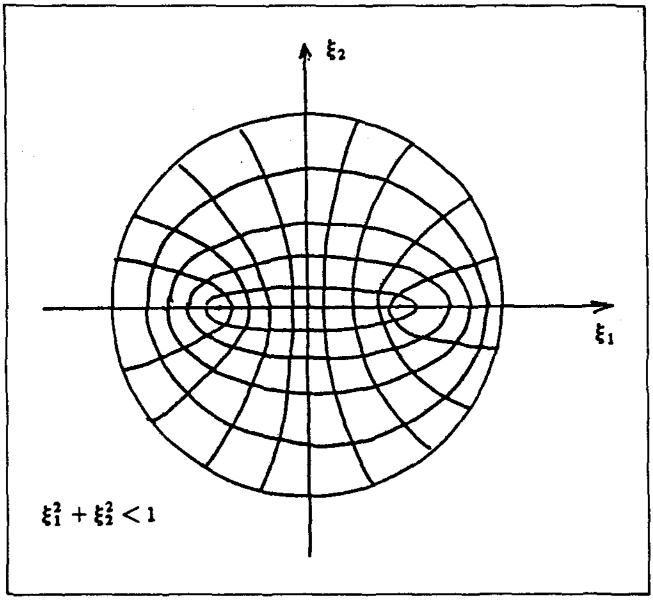

Fig. 4

$$\alpha_n < a_n < \alpha_{n-1} < a_{n-1} < \ldots < \alpha_1 < a_1 .$$

Proof. The graph of the left-hand side $\ell(\alpha)$ of equation (13) has the form, shown in the figure 3.

We use the fact that at infinity

$$|\alpha| \to \infty, \ \ell(\alpha) = -\frac{\langle x, x\rangle}{\alpha} = \frac{1}{\alpha} .$$

Now the proposition follows by continuity arguments. ■

The roots $\alpha_1, \ldots, \alpha_n$ correspond to different types of the H-conical sections, in particular α_n corresponds to H-ellipsoid passing through x.

Figure 4 shows the confocal family in the Klein model of the Lobachevsky plane.

The curves are the usual euclidean ellipses and hyperbolas, determined by

$$(14) \qquad \frac{\xi_1^2}{\left(\dfrac{a_1 - \alpha}{a_0 - \alpha}\right)} + \frac{\xi_2^2}{\left(\dfrac{a_2 - \alpha}{a_0 - \alpha}\right)} = 1,$$

for $\alpha = 0$ we have the initial H-ellipse (12).

The foci $F_\pm$ correspond to $\alpha = a_2$:

$$\xi_1 = \pm \sqrt{\frac{a_1 - a_2}{a_0 - a_2}}, \quad \xi_2 = 0.$$

The focal property of these points will follow from our results (see §4).

This family is orthogonal with respect to the Lobachevsky metric (but not the euclidean one). This fact can be proven in the same way as for the euclidean and spherical cases. For the completeness we give here the proof following Moser's arguments [7].

PROPOSITION 3. *The confocal surfaces (13) passing through the point $x \in \mathbb{R}^{n,1}$: $\langle x, x\rangle = -1$, $x_0 x_1 \ldots x_n \neq 0$, are orthogonal with respect to the metric $\langle\, , \rangle$ or equivalently the Lobachevsky metric.*

Proof. We prove that the normals to these surfaces coincide with eigenvectors of the selfadjoint operator

$$M = M_x = P_x A P_x$$

where $P_x = I + x \otimes x^*$ is the projector on the orthogonal complement of x (compare with [7]). One can check that

$$\det(M_x - \alpha I) = \alpha \det(A - \alpha I)\ \langle x, (A-\alpha)^{-1} x\rangle,$$

which means that the eigenvalues of M_x are $\alpha_0 = 0$ and $\alpha_1, \ldots, \alpha_n$, the latter being the roots of (13), i.e. the confocal coordiantes of x. The corresponding eigenvectors are $v_0 = x$ and $v_i = (A - \alpha_i)^{-1} x$, $i = 1, \ldots, n$. Indeed

$$(M - \alpha_i) v_i = (P_x A P_x - \alpha_i)\ v_i = P_x(A - \alpha_i) P_x v_i = P_x x = 0.$$

We use here that $P_x v_i = v_i$, which follows from the orthogonality x and v_i:

$$\langle v_i, x\rangle = \langle A - \alpha_i)^{-1} x, x\rangle = 0.$$

But the vectors $(A - \alpha_i)^{-1} x$ are the normals to the corresponding confocal surfaces. The proposition now follows from the orthogonality of the eigenvectors of the selfadjoint operator. ∎

Another way to prove this results is to calculate the metric tensor ds^2 in the coordinates $\alpha_1, \ldots, \alpha_n$. It can be done in the same way as for the sphere [2] and leads to the answer

$$ds^2 = \sum_{j=1}^{n} g_j(\alpha)\, d\alpha_j^2 .$$

$$g_j(\alpha) = \frac{1}{4} \frac{p'(\alpha_j)}{q(\alpha_j)},$$

where

$$p(z) = \prod_{i=1}^{n} (z - \alpha_i), \quad q(z) = \prod_{i=0}^{n} (z - a_i).$$

One can check that this formula implies the separation of variables for the Hamilton-Jacobi equation in these coordinates for the Hamiltonian functions $H = 1/2\, p^2$ and $H = 1/2\, p^2 + \mu\langle Ax, x\rangle$, corresponding to the geodesic flow on the Lobachevsky space and the hyperbolic analogue of C. Neumann's system. The last system, which can be viewed as an anisotropic harmonic oscillator in Lobachevsky space, is discussed in Appendix B.

2. THE H-ELLIPSOIDAL BILLIARD PROBLEM AND THE FACTORIZATIONS OF THE MATRIX POLYNOMIALS

In the paper [4] J. Moser and the author proposed the method for integration of discrete systems based on the factorization problem for the matrix polynomials. The idea can be explained in the example of the finite Toda lattice, which is connected with QR-algorithm as was discovered by W. Symes [12].

Recall this algorithm for finding the eivenvalues of the matrix [13]. The first step consists in the factorization of the given matrix $A = A_1$

$$A_1 = Q_1 R_1$$

where Q_1 and R_1 are the orthogonal and uppertriangular matrices correspondingly. If we suppose that A is nondegenerate and the diagonal elements of R_1 are positive, then such factorization is unique and given by the well-known Gram-Schmidt procedure.

On the second step we consider the new matrix

$$A_2 = R_1 Q_1$$

and its factorization

$$A_2 = Q_2 R_2.$$

Iterating this procedure we come to the sequence of matrices $A_1, A_2, \ldots$, which are similar: $RQ = Q^{-1}(QR)Q$. Under certain assumptions, for example, for symmetric matrices with different eigenvalues, this sequence has a limit, which is the diagonal form of the matrix $A = A_1$.

W. Symes proved that for a special initial matrix A this procedure corresponds to the Toda flow for integer time (see [12]). As was shown in [4] if we start from a certain quadratic matrix polynomial $L(\lambda)$

$$L(\lambda) = \ell_0 + \ell_1 \lambda + \ell_2 \lambda^2$$

and its factorization of the form

$$L(\Lambda) = (b_0 + b_1 \lambda)\,(c_0 + c_1 \lambda) = B(\lambda)\,C(\lambda),$$

then the analogous procedure

$$L(\lambda) \to L'(\lambda) = C(\lambda)\,B(\lambda) = C(\lambda)\,L(\lambda)\,C^{-1}(\lambda)$$

corresponds to the dynamics of the discrete versions of some classical integrable systems, in particular, the billiard dynamics in the ellipsoidal domain of the euclidean space.

Now we show that this method is working also in the Lobachevsky space (the case of the sphere can be considered in the same way).

Let x, y and z be the successive reflection points in the H-billiard (11):

$$\langle A^{-1}x, x\rangle = \langle A^{-1}y, y\rangle = \langle A^{-1}z, z\rangle = 0.$$

In the projective Klein model we have the straight lines xy and yz, which are in one plane with the normal N to the H ellipsoid (12) and form with N the angles, which are equal in the Lobachevsky metric (see fig. 5).

The main point of the factorization method is to find the corresponding matrix polynomial $L(\lambda)$. In our case $L(\lambda)$ turns out to be linear (!):

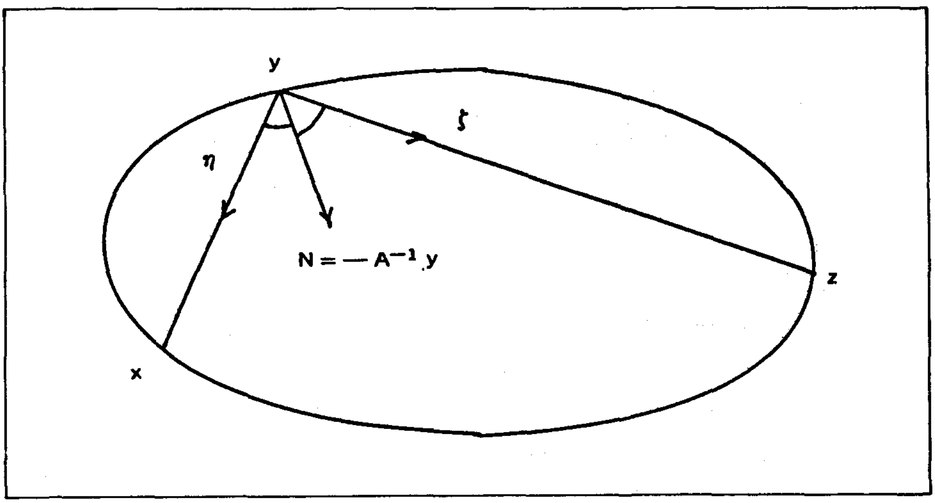

Fig. 5

$$L(\lambda) = A + \lambda(x \otimes y^* - y \otimes x^*). \tag{15}$$

This choice of $L(\lambda)$ is motivated by Moser's results [7].

Let's introduce the notations: the bivector

$$x \wedge y := x \otimes y^* - y \otimes x^* \tag{16}$$

is the skew symmetric operator in $V = \mathbb{R}^{n,1}$. The value

$$|x \wedge y|^2 := \langle x, y\rangle^2 - \langle x, x\rangle \langle y, y\rangle \tag{17}$$

is the area of the parallelogram generated by x and y in V.

Now we can start. Consider the factorization of $L(\lambda)$ (15) of the type

$$L(\lambda) = A + \lambda x \wedge y = (D + \lambda\eta \otimes \xi^*)(D - \lambda\xi \otimes \eta^*). \tag{18}$$

We have the relations

$$\begin{cases} D^2 = A \\ \eta \wedge D\xi = x \wedge y \\ \langle \xi, \xi\rangle = 0. \end{cases} \tag{19}$$

LEMMA. *The solutions of (19) after some transformation* $\xi \to \alpha\xi$, $\eta \to \alpha^{-1}\eta$ *have two possible forms:*

$$\xi = D^{-1}y, \quad \eta = x + \beta y \tag{20}$$

or

$$\xi = D^{-1}x, \quad \eta = -y + \beta x \tag{21}$$

with $D = \sqrt{A}$ *and artitrary* β.

Proof. The second equation of (19) says the vectors $D\xi$ and η generate the same plane Π as x and y. The third equation determines the cone which intersects with this plane in two lines, corresponding to (20) and (21). Indeed,

$$\langle \xi, \xi\rangle = \langle D^{-1}x, D^{-1}x\rangle = \langle A^{-1}x, x\rangle = 0.$$

It proves the lemma.

Two types of solutions, (20) and (21), correspond to two possible splittings of the roots of the scalar polynomial $P(\lambda) = \det L(\lambda)$:

$$\Sigma = \Sigma_1 \cup \Sigma_2 \tag{22}$$

where Σ, Σ_1 and Σ_2 are the sets of roots of the determinants of $L(\lambda)$, the first and second factors correspondently (compare with [4])

In our case

(23) $$P(\lambda) = \det(A + \lambda x \wedge y) = \det A(1 - \lambda^2 \langle A^{-1}x, y\rangle^2)$$

and we have two possibilities

(24) $$\Sigma_1 = -\langle A^{-1}x, y\rangle^{-1}, \qquad \Sigma_2 = +\langle A^{-1}x, y\rangle^{-1}$$

and

(25) $$\Sigma_1 = \langle A^{-1}x, y\rangle^{-1}, \qquad \Sigma_2 = -\langle A^{-1}x, y\rangle^{-1},$$

which correspond to (20) and (21) respectively.

Indeed, for ξ and η given by (20), for instance, we have

$$\det(D - \lambda\xi \otimes \eta^*) = \det D(1 - \lambda \langle D^{-1} \xi, \eta\rangle) =$$
$$= \det D(1 - \lambda \langle A^{-1}x, y\rangle).$$

Now we make a choice, which corresponds to the direction of the motion from x to y, fixing the splitting (24).

For convenience we choose the corresponding ξ and η in (20) in a such way, that η is orthogonal to y, i.e.

(26) $$\beta = -\frac{\langle x, y\rangle}{\langle y, y\rangle}.$$

Changing the order of the factors in (24) we come to

(27) $$L'(\lambda) = (D - \lambda\xi \otimes \eta^*)(D + \lambda\eta \otimes \xi^*) =$$
$$= A + \lambda D\eta \wedge \xi - \lambda^2 \langle \eta, \eta\rangle \xi \otimes \xi^*.$$

We see that the first step changed the degree of the matrix polynomial L. Make one more step:

(28) $$L'(\lambda) = (D + \lambda\xi \otimes \zeta^*)(D - \lambda\zeta \otimes \xi^*) =$$
$$= A + \lambda\xi \wedge D\zeta - \lambda^2 \langle \zeta, \zeta\rangle \xi \otimes \xi^*.$$

The comparison of (27) and (28) leads to

(29) $$\begin{cases} \langle \zeta, \zeta\rangle = \langle \eta, \eta\rangle \\ D\zeta + D\eta = \nu\xi. \end{cases}$$

The last equation can be rewritten as

(30) $$\zeta + \eta = \nu D^{-1}\xi = \nu A^{-1} y.$$

Together with the first equation it leads to two solutions

$$\zeta = -\eta$$

and

$$(31) \qquad \zeta = -\eta + \nu A^{-1} y\,, \quad \nu = \frac{2\langle \eta_0 A^{-1} y\rangle}{\langle A^{-1}y, A^{-1}y\rangle}$$

One can easily check that only the second possitility corresponds to our splitting (24) and to the direction of the billiard particle after the reflection in the point y (see fig. 5).

To finish the step let us find

$$(32) \qquad L''(\lambda) = (D - \lambda\zeta \otimes \xi^*)(D + \lambda\xi \otimes \zeta^*) = A + \lambda D\xi \wedge \zeta = A + \lambda y \wedge \zeta,$$

where $z = \zeta + \gamma y \in M^n$ is uniquely determined by the equation

$$(33) \qquad \langle A^{-1}z, z\rangle = 0 : \gamma = -\frac{\langle A^{-1}\xi, \zeta\rangle}{\langle A^{-1}y, \zeta\rangle}$$

Thus after two steps our procedure leads to the transformation $L(\lambda) \to L''(\lambda)$, which corresponds to the billiard dynamics $(x, y) \to (y, z)$ ■

Notice that the vector z determined by (33) satisfies the relation

$$|y \wedge z|^2 = |x \wedge y|^2.$$

It determines the correct normalization for the vector in the projective model of the Lobachevsky space.

We summarize the results of this section in the following theorem.

THEOREM 1. *Let $\{x_k\}$ be an orbit in the billiard problem in the H-ellipsoidal domain of Lobachevsky space, which in projective representation in $\mathbb{R}^{n,1}$, determined by the equation $\langle Ax, x\rangle \leqslant 0$. Choose the vectors x_k in a such way that* $|x_k \wedge x_{k+1}|^2 = \mathrm{const}$. *Then the matrix*

$$L_k = A + \lambda x_{k-1} \wedge x_k$$

undergoes the isospectral transformation

$$(34) \qquad L_{k+1} = A_k L_k A_k^{-1}$$

where

$$(35) \qquad A_k = A - \lambda(\zeta_k \otimes x_k^* + x_k \otimes \eta_k^*),$$

ξ_k and η_k are tangent vectors to the trajectory at the reflection point x_k (see fig. 5) and are defined by the formula (26). Moreover, L_{k+1} is the result of two steps of the factorization procedure, described before and applied to L_k.

The relations (34), (35) follow from the previous considerations but can be checked also by the straightforward calculation.

COROLLARY. *The H-ellipsoidal billiard problem has the following integrals* F_j:

(36) $$F_j = \sum_{i \neq j} \frac{J_i J_j (x_i y_j - x_j y_i)^2}{A_j - A_i} \qquad (j = 0, 1, \ldots, n)$$

which satisfy the unique relation

$$F_0 + F_1 + \ldots + F_n = 0.$$

Recall that for $\mathbb{R}^{n,1}$, $J_0 = -1, J_1 = J_2 = \ldots = J_n$.

The corollary follows from the theorem 1 and the formula

(37) $$\det (L - \mu I) = \det (A - \mu I)\,(1 - \lambda^2 \phi_\mu(x, y)),$$

where

(38) $$\phi_\mu(x, y) = \langle (A - \mu I)^{-1} x, y\rangle^2 - \langle (A - \mu I)^{-1} x, x\rangle \langle (A - \mu I)^{-1} y, y\rangle = \sum_{i=0}^{n} \frac{F_i}{A_i - \mu}.$$

One can show using [2, 7] that these integrals are in involution with respect to the natural symplectic structure, which can be defined in the hyperbolic case in the same way as for the euclidean case (see for example [3]). Therefore, this billiard problem is integrable in Liouville sense [3] and corresponds to the translations on the Liouville tori. In the next section we make the last sentence more precise.

3. HYPERELLIPTIC CURVES AND θ-FUNCTIONS

We show now how one can use the results of the previous section for the explicit integration of the billiard dynamics in an H-ellipsoidal domain. We will follow the algebrogeometrical method of "finite-gap" integration, which for the matrix systems was developed by Dubrovin (see [14, 15]).

Let us consider the spectral curve Γ, determined by the equation

(39) $$\det (L(\lambda) - \mu I) = \det (A + \lambda x \wedge y - \mu I) = 0.$$

By using (37), (38) one can rewrite it as

(40) $$p(\mu) - \lambda^2 \,|x \wedge y|^2\, q(\mu) = 0$$

with

$$p(\mu) = \prod_{i=0}^{n} (\mu - a_i), \quad q(\mu) = \prod_{i=1}^{n} (\mu - \mu_i), \ \mu_i, \ldots, \mu_n$$

are the roots of $Q_\mu(x, y)$:

$$\sum_{k=0}^{n} \frac{F_k(x, y)}{a_k - \mu} = 0. \tag{41}$$

Let $A_0, \ldots, A_n$ be the points of Γ with $\lambda = 0$ and $\mu = a_0, \ldots, \mu = a_n$ correspondently, $P_\pm$ be the "infinities": $\mu \approx \pm \lambda \,|\, x \wedge y \,|, \lambda \to \infty$.

The eigenvector ψ of $L(\lambda)$:

$$(L(\lambda) - \mu I)\, \psi(\lambda, \mu) = 0 \tag{42}$$

normalized by the condition

$$\psi^0 + \ldots + \psi^n = 1 \tag{43}$$

is the meromorphic vector-function on Γ with pole-divisor $\mathscr{D}$ of degree $(n + 1) + g - 1 = 2n - 1$, where $g = (n - 1)$ is the genus of Γ (see [14, 15]).

The meromorphic functions with the pole-divisor $\leqslant \mathscr{D}$ form the vector space $L(\mathscr{D})$ of the dimension $(n + 1)$ for the general $\mathscr{D}$. The coordinates $\psi_0, \ldots, \psi_n$ form the basis of $L(\mathscr{D})$, uniquely determined by the conditions

$$\psi_i(A_j) = \delta_{ij} \tag{44}$$

which follow from (42), (43).

The knowledge of $\psi(\lambda, \mu)$ allows to reconstruct the bivector $x \wedge y$ by the formula

$$x \wedge y = \alpha \psi\,(P_+) \wedge \psi(P_-), \tag{45}$$

α is determined from $|\, x \wedge y \,| = c$. Indeed, for large λ the eigenfunction ψ is close to the eigenfunctions of the matrix $x \wedge y$: $(x \wedge y)\, \phi = \nu\phi$ for the nonzero eigenvalues $\nu = \pm \,|\, x \wedge y \,|$. Choose the basis e_1, e_2 in the plane M_{xy} in such a way that $\langle e_1, e_1 \rangle = 1$, $\langle e_2, e_2 \rangle = -1$, $\langle e_1, e_2 \rangle = 0$. Then the eigenvectors $\phi_\pm$, corresponding to $\nu = \pm \,|\, x \wedge y \,|$ are given by

$$\phi_\pm = e_1 \pm e_2.$$

It leads to the formula (45).

Let us now return to theorem 1. As follows from (34), (35) the eigenvector ψ_{k+1} of L_{k+1} can be expressed through ψ_k by the formula

(46) $$\psi_{k+1} = A_k \psi_k = (A - \lambda(\zeta_k \otimes x_k^* + x_k \otimes \eta_k^*)) \psi_k.$$

We see that ψ_{k+1} has two new poles in P_+ and P_- ($\lambda = \infty$) and the new double zero in the point Q_+, corresponding to
$\mu = 0, \lambda = \langle x, A^{-1}y\rangle^{-1} = \langle x_{k-1}, A^{-1}x_k\rangle^{-1}$ (see (23)). Indeed,

$$\det A_k = \det A(1 - \lambda \langle x_{k-1}, A^{-1} x_k\rangle)^2,$$

which follows immediately from the factorization procedure.

It means that the pole-divisor $\mathscr{D}_{k+1}$ of the vector function ψ_{k+1} differs from $\mathscr{D}_k$ up to the linear equivalence on the divisor U

(47) $$\mathscr{D}_{k+1} = \mathscr{D}_k + U,$$

where $U = P_+ + P_- - 2Q_+ = Q_- - Q_+$, Q_- has the coordinates $\mu = 0, \lambda = -\langle x, A^{-1}y\rangle^{-1}$. The equivalence $P_+ + P_- = Q_+ + Q_-$, which we use, is given by the function $f(\mu, \lambda) = \mu$.

We complete our considerations exhibiting the formulas for ψ in terms of θ-functions. All necessary facts about θ-functions can be found in [16], (see also [17], where the application of these functions are discussed).

Let $\omega_1, \ldots, \omega_g$ be the basis of the holomorphic 1-forms on Γ, normalized by the conditions

$$\oint_{\alpha_k} \omega_j = 2\pi i \delta_{jk}$$

for some canonical basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ in $H^1(\Gamma, \mathbb{Z})$. The Jacobian variety $J(\Gamma)$ is determined as

$$J(\Gamma) = \mathbb{C}^g/L$$

where the lattice L is generated by the columns of the matrices $2\pi iI$ and B, B is the Riemann matrix:

$$B_{kj} = \oint_{\beta_k} \omega_j.$$

The classical Riemann theta-function $\theta(z)$ is determined as the series

$$\theta(z) = \theta(z \mid B) = \sum_{m \in \mathbb{Z}^g} \exp\left(\frac{1}{2} m\, Bm^t + zm^T\right).$$

Let $\mathscr{A} : \Gamma \to J(\Gamma)$ be the Abel mapping

$$\mathscr{A}(P)^i = \int_{P_0}^{P} \omega_i$$

for some fixed point $P_0 \in \Gamma$. The function $f(P) = \theta(\mathscr{A}(P) - \zeta)$ for the general $\zeta \in J(\Gamma)$ has g zeroes $P_1, \ldots, P_g$ on Γ with the property

$$\mathscr{A}(P_1) + \ldots + \mathscr{A}(P_g) = \zeta + K,$$

where K is the vector of Riemann constants [16].

Let's consider the function $\psi^0(P)$. It has $(2n - 1)$ poles and the same number of zeroes, from which n coincides with $A_1, \ldots, A_n$. The remaining zeroes we denote as $Z_1, \ldots, Z_{n-1}$.

The ratio $\psi^1(P)/\psi^0(P)$ has $n = g + 1$ poles $A_1, Z_1, \ldots, Z_{n-1}$ on Γ and one of the zeroes at the point A_0. This property determines this function up to a constant factor.

By using the transformation law of $\theta(z)$, $\theta(z + 2\pi i\, m + Bn) = \exp(-1/2\; nBn^T - zn^T)\, \theta(z)$ one can check that the following function has the same property

$$\phi_{01}(P) = \exp\left(\int_{P_0} \Omega_{01}\right) \frac{\theta(\mathscr{A}(P) + V_{01} - \zeta)}{\theta(\mathscr{A}(P) - \zeta)} \tag{48}$$

Here Ω_{01} is the meromorphic 1-form with the poles at A_0 and A_1 with the residues $(+1)$ and (-1) correspondently and with zero α-periods: V_{01} is the vector of its β-periods:

$$V_{01}^i = \oint_{\beta_i} \Omega_{01} = \mathscr{A}(A_0) - \mathscr{A}(A_1),\ \zeta = \mathscr{A}(z_1) + \ldots + \mathscr{A}(z_{n-1}) - K.$$

Thus we have the formula for $\psi_k(P)$:

$$\psi_k(P) = (\theta(A(P) - \zeta_k) : c_1\, \phi_1(P)\, \theta(A(P) - V_{01} - \zeta_k) : \ldots \tag{49}$$
$$\ldots : c_n\, \phi_n(P)\, \theta(A(P) - V_{0n} - \zeta_k))$$

where

$$\phi_i(P) = \exp\left(\int_{P_0}^{P} \Omega_{0i}\right), \quad \zeta_k = \zeta_0 + kU, \quad \zeta_0 \in J(\Gamma)$$

and the constants $c_1, c_2, \ldots, c_n$ are determined by the initial data, U is the same as in (47), $V_{0i} = \mathscr{A}(A_0) - \mathscr{A}(A_i)$ are the half-periods.

THEOREM 2. *The dynamics in the H-ellipsoidal billiard problem corresponds to the shift (47) on the Jacobi variety of the hyperelliptic curve (40) and is given in terms of θ-functions by the formulas (45), (49).* ■

4. GEOMETRICAL CONSEQUENCES

We begin with the following theorem, generalizing a well-known fact of euclidean geometry [1].

THEOREM 3. *All sides of the trajectory in the H(S)-ellipsoidal billiard problem in the Lobachevsky space (on the sphere) are tangent to the $(n - 1)$ confocal conic sections, which are fixed for a given trajectory.*

Proof. We will consider only the hyperbolic case. As follows from the formulas (34) - (38) the roots of the function

$$\phi_\mu(x_{k-1}, x_k) = 0$$

do not depend on k for *and* billiard trajectory x_k. We need that only in following lemma.

LEMMA. *For given points x and y in $\mathbb{R}^{n,1}$ with $\langle x, x\rangle = \langle y, y\rangle = -1$ the equation $\phi_\mu(x, y) = 0$:*

$$\langle (A - \mu I)^{-1} x, y\rangle^2 - \langle (A - \mu I)^{-1} x, x\rangle \langle (A - \mu I)^{-1} y, y\rangle = 0$$

has $(n - 1)$ real roots $\mu_1, \ldots, \mu_{n-1}$. The straight line xy in the Lobachevsky space is tangent to the confocal conical sections (7), corresponding to $\alpha = \mu_1, \ldots, \alpha = \mu_{n-1}$.

Proof of the lemma. We essentially repeat the arguments of Moser and Arnold for the analogous euclidean situation.

The function $\phi_\mu(x, y)$ is the discriminant of the quadratic equation on t:

$$\langle (A - \mu I)^{-1} (x + ty), \quad (x + ty)\rangle = 0,$$

describing the intersection of the straight line xy with the confocal surface (13), $\alpha = \mu$. Therefore $\phi_\mu(x, y) = 0$ describes the conical sections, which are tangent to the given line xy. Choose an orthogonal basis e_1, e_2 in the plane M_{xy} in $\mathbb{R}^{n,1}$: $\langle e_1, e_2\rangle = -1$, $\langle e_2, e_2\rangle = 1$, $\langle e_1, e_2\rangle = 0$. The orthogonal projection of $V = \mathbb{R}^{n,1}$ on W which is the orthogonal complement of e_2 : $\langle W, e_2\rangle = 0$, transforms the confocal family (13) into the new confocal family in W, which is supplied by the induced pseudoeuclidean structure. Indeed, for the dual surfaces it leads to the restriction on W, which obviously conserves the property of the pseudoeuclidean pensils (6).

Thus one can use the proposition (2) to complete the proof of the lemma and of theorem 3. ∎

Example 1. H-elliptical billiard on the Lobachevsky plane (see fig. 4 and 6). The picture in the Klein model is very similar to the euclidean one, the only difference is in the definition of the confocal ellipses (14).

COROLLARY 1. (Focusing property of the H-elliptical billiard).

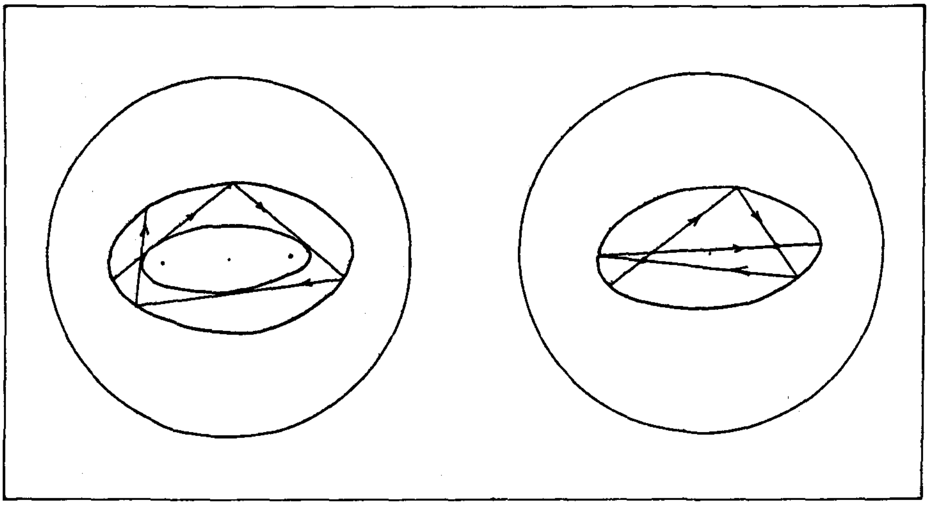

Fig. 6

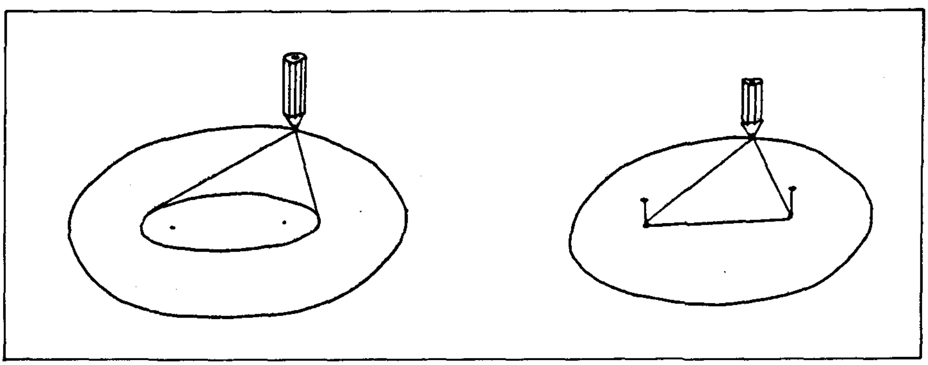

Fig 7

The trajectories, passing through one of the foci $F_\pm$ after the reflection come into another focus (see fig. 6). ■

Another consequence generalizes the string construction of the euclidean ellipse, given by Graves (see [1, 10]).

GRAVES THEOREM. *Given an ellipse E and a closed piece of string with length strictly greater than the length of C, the locus of a pencil used to pull the string taut around C is another ellipse C', confocal with C (see fig. 7).* ■

COROLLARY 2. *Graves theorem is true also in the Lobachevsky plane as well as on the sphere for the corresponding conical sections. In particular, the H-ellipse can be determined geometrically as the locus of X*

$$\rho(X, F_-) + \rho(X, F_+) = \text{const.}$$

where ρ is Lobachevsky distance.

It follows from the calculation of the first variation of the string's length and the reflection property. ■

The three-dimensional euclidean generalization of Graves construction was found by O. Staude [18]. It seems to be possible to prove the analogous theorem in the Lobachevsky space.

Example 2. *S-elliptical billiard on the sphere $S^2 \subset \mathbb{R}^3$ (see also* [10], *p. 312).*

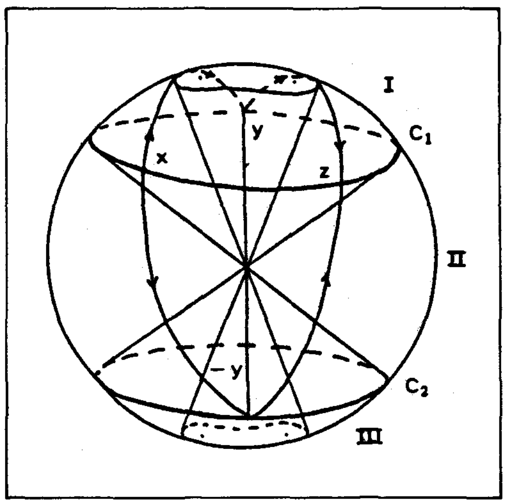

Fig. 8

The cone

$$\frac{x_0^2}{a_0} + \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} = 0$$

$(a_0 > a_1 > 0 > a_2)$ intersects with the unit sphere S^2 by the closed curves C_1 and C_2, dividing S^2 into three domains I, II and III. After the projectivization $S^2 \to \mathbb{R}P^2$ we have two domains I and II, homeomorphic to the disc and the Möbius sheet correspondently. The billiard dynamics in these two domains coincide on their common boundary C (see fig. 8). It is interesting to note that the domain II is concave everywhere. in the zero curvature situation this property leads to the stochastic behaviour, as the famous example of Ya. G. Sinai [19] shows.

APPENDIX A

The geodesic flow on the H ellipsoid in Lobachevsky space

The geodesic flow on the H-ellipsoid can be considered as the continuous limit of the corresponding billiard system. Therefore its integrability in some sense follows from our results. To make it more convincing we present here the Lax representation with the spectral parameter, which is crucial for the complete integrability.

Let $x(s)$ be the geodesic on the H-ellipsoid, determined by the equation

$$(1) \qquad -\frac{x_0^2}{a_0} + \frac{x_1^2}{a_1} + \ldots + \frac{x_n^2}{a_n} = 0,$$

$x \in \mathbb{R}^{n,1}$, $\langle x, x\rangle = -1$ (see § 1). Then $x(s)$ satisfies the equation

$$(2) \qquad \ddot{x} = \nu A^{-1}x + \mu x,$$

$A = \operatorname{diag}(a_0, a_1, \ldots, a_n)$, $\mu = 1$, $\nu = \nu(x)$ is determined from the conditions

$$\langle x, A^{-1}x\rangle = 0,\ \langle x, x\rangle = -1:\ \langle \dot{x}, A^{-1}x\rangle =$$
$$= 0 \to \langle \ddot{x}, A^{-1}x\rangle + \langle \dot{x}, A^{-1}\dot{x}\rangle = 0$$

and therefore

$$(3) \qquad \upsilon = -\frac{\langle \dot{x}, A^{-1}\dot{x}\rangle}{\langle A^{-1}x, A^{-1}x\rangle}$$

THEOREM. *The equation (2), (3) for the geodesics on the H-ellipsoid is equivalent to the Lax representation with the spectral parameter* λ

(4) $$\dot{L} = [L, P]$$

where $L = A + \lambda x \wedge \dot{x}$ $P = \lambda \nu A^{-1} x \otimes (A^{-1} x)^*$.

Recall (see § 1) that for $x, y \in \mathbb{R}^{n,1}$, $x \wedge y = x \otimes y^* - y \otimes x^*$.

The theorem can be proved by a simple calculation.

Now one can find the explicit formulas in terms of theta-functions of the hyperelliptic curve Γ:

$$\det(L(\lambda) - \mu I) = 0$$

in the same way as it was done for the billiard system in § 3. The corresponding formulas for the geodesics on the euclidean ellipsoid in $\mathbb{R}^3$ were found by K. Weierstrass [20], for higher dimension it was done by H. Knörrer [21] (see also [22]).

APPENDIX B

Harmonic oscillator in the Lobachevsky space and finite-gap Schrödinger operators

We will consider the hyperbolic version of C. Neumann system, which can be viewed as a natural analogue of the anisotropic harmonic oscillator in the Lobachevsky space.

We use the notations of § 1. Let x be the points of one sheet of the hyperboloid in $\mathbb{R}^{n,1}$

(1) $$\langle x, x \rangle = -x_0^2 + x_1^2 + \dots + x_n^2 = -1.$$

The hyperbolic version of C. Neumann system [6] corresponds to the Hamiltonian function

(2) $$H = \frac{1}{2} \langle p, p \rangle + \frac{1}{2} \langle Ax, x \rangle,$$

$A = \operatorname{diag}(a_0, \dots, a_n)$, $a_0 < a_1 < \dots < a_n$. The potential $U(x) = 1/2 \langle Ax, x \rangle$ in the coordinates $x_1, \dots, x_n$ has the form

(3) $$U = \frac{1}{2} (-a_0 (x_1^2 + \dots + x_n^2 + 1) + a_1 x_1^2 + \dots + a_n x_n^2) =$$

$$= \frac{1}{2} [(a_1 - a_0) x_1^2 + \dots + (a_n - a_0) x_n^2 - a_0],$$

which make evident the analogy with the usual harmonic oscillator.

The equations of the motion are

$$\ddot{x} + Ax = \lambda x \tag{4}$$

with the Lagrange multiplyer

$$\lambda = \langle Ax, x\rangle - \langle \dot{x}, \dot{x}\rangle$$

In order to integrate this system one can use the Lax representation, which is a slight modification of that found by J. Moser [7].

PROPOSITION. *The system (4) has the Lax representation with the spectral parameter μ*

$$\dot{L} = [L, P]$$

with $L = A + \mu x \wedge \dot{x} + \mu^2 x \otimes x^* \quad p = \mu x \otimes x^*$

Another possibility is supplied by the connection of this problem with the spectral theory of the Schrödinger operator

$$\mathscr{L} = -\frac{d^2}{dt^2} + u(t). \tag{5}$$

discovered by J. Moser and E. Trubowitz [2, 8].

The finite-gap theory of this operator which began with the pioneering paper of S.P. Novikov [23] can be found in [14].

Let $u(t)$ be the n-gap operator $\mathscr{L}$ with the spectrum

$$[E_0, E_1] \cup [E_2, E_3] \cup \ldots \cup [E_{2n}, \infty].$$

The Bloch eigenfunction ψ

$$\mathscr{L}\psi = E\psi$$

is meromorphic on the affine part of the hyperelliptic curve Γ:

$$y^2 = \prod_{i=0}^{2n} (E - E_i) = R(E)$$

and has n poles. Its projections on the line $E - \gamma_1, \ldots, \gamma_n$ belong to the gaps: $E_{2i-1} < \gamma_i < E_{2i}$.

Let $a_0 < \ldots < a_n$ be any $(n + 1)$ points from the set $E_0, \ldots, E_{2n}$.

LEMMA. (see [9]). *The functions $\psi(a_\alpha, t)$ satisfy the identity*

$$\sum_{\alpha=0}^{n} c_\alpha \psi^2(a_\alpha, t) \equiv 1 \tag{6}$$

with

$$(7) \qquad c_\alpha = \frac{\prod_{j=1}^{n} (a_\alpha - \gamma_j)}{\prod_{\beta \neq \alpha} (a_\alpha - a_\beta)}$$

The proof follows from the formula for $\psi^2(E, t)$ in the points $E = E_i$:

$$(8) \qquad \psi^2(E_i, t) = \prod_{j=1}^{n} \frac{E_i - z_j(t))}{(E_i - \gamma_j)},$$

proved in [14]. ■

Suppose we have the inequalities

$$(9) \qquad c_0 > 0, \; c_i < 0 \quad (i = 1, \ldots, n).$$

Then the formulas

$$(10) \qquad \begin{aligned} x_0(t) &= \sqrt{+c_0}\, \psi(a_0, t) \\ x_i(t) &= \sqrt{-c_i}\, \psi\ (a_i, t), \quad i = 1, \ldots, n \end{aligned}$$

determine the solution of the system (4). Indeed,

$$\mathscr{L}x = -\ddot{x} + u(t)\,x = Ax$$

and $\langle x, x\rangle \equiv -1$ from (7) and (10).

One can check that the inequalities (9) are equivalent to the following choice of $a_0, \ldots, a_n$, which we will call *admissible:* $a_0 = E_0$, $a_1 = E_1$ and between a_i and a_{i+1} ($(i \geqslant 1)$ lies exactly one gap (see fig. 9 (H))

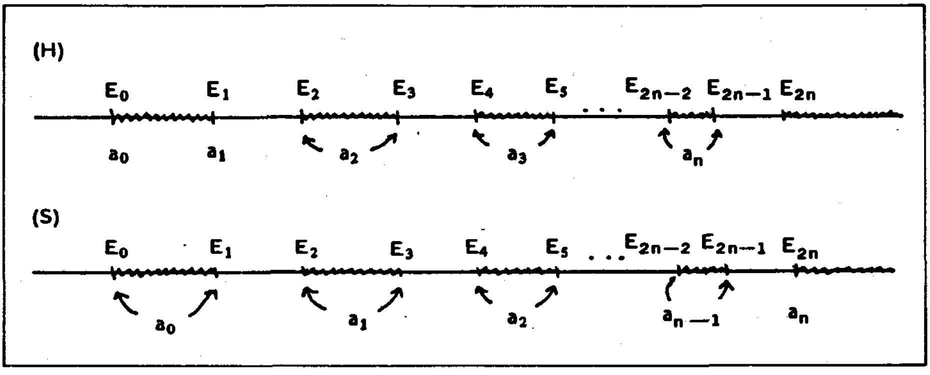

Fig. 9

For the comparison we show also in fig. 9 (S) the choice of a_i, corresponding to the usual C. Neumann system (see [22]).

THEOREM. *Let $\psi(E, t)$ be the Bloch eigenfunction of the n-gap Schrödinger operator $\mathscr{L}$, let $a_0, \ldots, a_n$ be an admissible subset of the edges of its spectrum $E_0, \ldots, E_{2n}$. Then the formulas (10), (7) determine the general solution of the system (4), describing the harmonic oscillations in the Lobachevsky space.* ■

This theorem combined with the results of [14] leads to the following formulas for the solutions:

$$x_\alpha(t) = x_\alpha(0)\,\frac{\theta_\alpha(tU+\zeta)\,\theta(\zeta)}{\theta(tU+\zeta)\,\theta_\alpha(\zeta)} \tag{11}$$

Here $\theta_\alpha(z)$ are theta-functions on the $J(\Gamma)$ with the characteristics corresponding to the second order points a_α (see [16]), U is the vector of b-period of the normalized abelian differential Ω with the pole of the second order at infinity (see [14, 17])

For the system on the sphere S^n for $n = 2$ such formulas were found by C Neumann [6] for other n it was done by the author in [22] (see also [17]).

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Note added in the proof. Recently H. Knorrer acquainted me with the paper [24], which also discuss the billiards on the sphere and pseudosphere. The problem of the explicit integration of the corresponding discrete systems is not considered in this paper.

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